

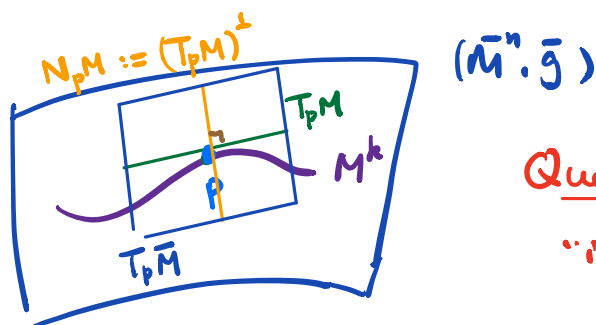
MATH 5061 Lecture 8 (Mar 10)

Recall: Isometric immersion $F: (M^k, g) \rightarrow (\bar{M}^n, \bar{g})$ st $F^* \bar{g} = g$

$$\text{i.e. } \bar{g}_{F(p)}(dF_p(v), dF_p(w)) = g_p(v, w)$$

Locally, immersions are embedding, $M^k \simeq F(M) \subseteq \bar{M}^n$.

Setup: $M^k \subseteq (\bar{M}^n, \bar{g})$ submanifold ($k \leq n$)



Question: How do study the "intrinsic" & "extrinsic" geometry of M ?

Crucial observation: At $p \in M$, there is an

orthogonal splitting (w.r.t. \bar{g}) $T_p \bar{M} = T_p M \oplus_{\perp} (T_p M)^{\perp}$

Notation: $NM := \coprod_{p \in M} (T_p M)^{\perp}$ normal bundle

orthogonal decomposition $V = \underbrace{V^T}_{T_p M} + \underbrace{V^N}_{N_p M} \quad \forall p \in M$

Note: \bar{g} restricts to an inner product on each $T_p M \subseteq (T_p \bar{M}, \bar{g})$

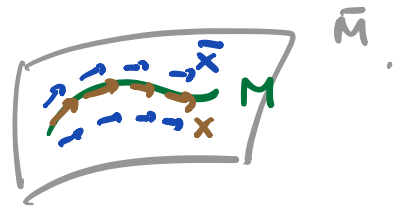
write $g := \bar{g}|_{TM} \rightsquigarrow (M^k, g)$ Riem. manifold

By Fund. Thm. of R.G., $\exists!$ Riem. connection ∇ for (M^k, g) .

Q: How is ∇ related to the ambient Riem. connection $\bar{\nabla}$ on (\bar{M}, \bar{g}) ?

Recall: $\nabla : T(TM) \times T(TM) \rightarrow T(TM)$

$\bar{\nabla} : T(T\bar{M}) \times T(T\bar{M}) \rightarrow T(T\bar{M})$



Prop: Let $X, Y \in T(TM)$, and $\bar{X}, \bar{Y} \in T(T\bar{M})$ be extensions of X, Y .

THEN:

$$\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^T$$

Remarks: $\bar{\nabla}_{\bar{X}} \bar{Y}(p)$ depends only on $\bar{X}(p) = X(p)$

and \bar{Y} along any curve $\gamma : (-\epsilon, \epsilon) \rightarrow \bar{M}$ s.t. $\gamma(0) = p, \gamma'(0) = X(p)$

[we can make $\gamma \subseteq M$, where $\bar{Y} = Y$ on M]

\Rightarrow R.H.S. is indep. of the choice of extensions \bar{X}, \bar{Y} .

Proof: Check R.H.S. defines a connection which is metric compatible and torsion^{-free}, then result follows by uniqueness part of Fund. Thm of R.G.

(i) $(X, Y) \mapsto (\bar{\nabla}_X Y)^T$ bilinear

(ii) $(\bar{\nabla}_{fX} Y)^T = (f \bar{\nabla}_X Y)^T = f (\bar{\nabla}_X Y)^T$

(iii) $(\bar{\nabla}_X (fY))^T = (X(f)Y + f \bar{\nabla}_X Y)^T = X(f)Y + f (\bar{\nabla}_X Y)^T$

(iv) $Z(g(X, Y)) = \bar{Z}(\bar{g}(\bar{X}, \bar{Y})) = \bar{g}(\bar{\nabla}_{\bar{Z}} \bar{X}, \bar{Y}) + \bar{g}(X, \bar{\nabla}_{\bar{Z}} \bar{Y})$
 $= g((\bar{\nabla}_{\bar{Z}} \bar{X})^T, Y) + g(X, (\bar{\nabla}_{\bar{Z}} \bar{Y})^T)$

(v) $(\bar{\nabla}_{\bar{X}} \bar{Y})^T - (\bar{\nabla}_{\bar{Y}} \bar{X})^T = (\bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{Y}} \bar{X})^T = ([\bar{X}, \bar{Y}])^T$
 $= ([X, Y])^T = [X, Y]$

Connection.

metric-compatible

torsion-free

Q: What about the "normal" part of $\bar{\nabla}$?

Defⁿ: 2nd fundamental form of M in \bar{M}

$$A : T(TM) \times T(TM) \rightarrow T(NM)$$

$$A(X, Y) := (\bar{\nabla}_{\bar{X}} \bar{Y})^N$$

Remark: This is well-defined indep. of the extensions \bar{X}, \bar{Y} .

Lemma: (i) $A(X, Y) = A(Y, X)$

$\forall X, Y \in T(TM)$
 $\forall f \in C^\infty(M)$

$$(ii) A(fX, Y) = A(X, fY) = f A(X, Y)$$

ie. A is a symmetric NM -valued (0,2)-tensor.

Proof: (i) $A(X, Y) - A(Y, X) = (\bar{\nabla}_{\bar{X}} \bar{Y})^N - (\bar{\nabla}_{\bar{Y}} \bar{X})^N$
 $= (\bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{Y}} \bar{X})^N = ([\bar{X}, \bar{Y}])^N = ([X, Y])^N = 0$

(Because $[X, Y] \in T(TM) \forall X, Y \in T(TM)$).

$$(ii) A(fX, Y) = (\bar{\nabla}_{\bar{f}\bar{X}} \bar{Y})^N = (\bar{f} \bar{\nabla}_{\bar{X}} \bar{Y})^N = f (\bar{\nabla}_{\bar{X}} \bar{Y})^N = f A(X, Y).$$

Fix $\eta \in T(NM)$, then we can define a scalar-valued 2nd ff. (w.r.t η)

$$A_\eta : T(TM) \times T(TM) \rightarrow C^\infty(M)$$

$$A_\eta(X, Y) = \langle A(X, Y), \eta \rangle$$

This is a symmetric bilinear form on each $T_p M$.

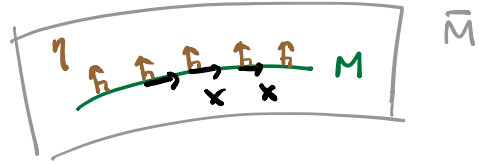
linear
algebra

$$A_\eta(X, Y) = \langle S_\eta(x), Y \rangle$$

shape operator /
Weingarten map

where $S_\eta : T_p M \rightarrow T_p M$ is self-adjoint operator.

Prop: $S_\eta(x) = -(\bar{\nabla}_x \eta)^T$



Proof: $\langle S_\eta(x), Y \rangle = A_\eta(x, Y) = \langle A(x, Y), \eta \rangle$
 $= \langle (\bar{\nabla}_x \bar{Y})^N, \eta \rangle = \langle \bar{\nabla}_x \bar{Y}, \eta \rangle$ $\forall x, Y \in T(M)$
 $= \underbrace{X \langle Y, \eta \rangle}_{=0} - \langle Y, \bar{\nabla}_x \eta \rangle = \langle Y, -(\bar{\nabla}_x \eta)^T \rangle$

Question:

$(\bar{M}^n, \bar{g}) \rightsquigarrow$ connection $\bar{\nabla}$ \rightsquigarrow Curvature \bar{R}
 \cup related \downarrow \uparrow relation?
 $(M^k, g) \rightsquigarrow$ connection ∇ \rightsquigarrow curvature R

Answer: \bar{R} and R are related via the 2nd f.f. A (or S_η)

They will be expressed in terms of 3 sets of "constraint equations" called Gauss, Codazzi, Ricci equations.

Before we state these equations, we need some preliminary notions:

(i) | Connection & curvature on normal bundle NM

\exists connection ∇^\perp on NM defined as:

$\nabla^\perp : T(TM) \times T(NM) \rightarrow T(NM)$

$\nabla_x^\perp \eta := (\bar{\nabla}_x \eta)^N$

(Ex: This is a "connection")

\rightsquigarrow normal curvature $R^\perp(x, Y) \eta := \nabla_Y^\perp \nabla_x^\perp \eta - \nabla_x^\perp \nabla_Y^\perp \eta + \nabla_{[x, Y]}^\perp \eta$

(ii) Covariant derivative of 2nd ff. A

Fix $\eta \in T(NM)$, then

$$A_\eta(x, Y) = \langle A(x, Y), \eta \rangle =: A(x, Y, \eta)$$

we regard $A : T(TM) \times T(TM) \times T(NM) \rightarrow C^\infty(M)$

Define: $\forall X, Y, Z \in T(TM), \forall \eta \in T(NM)$.

$$\begin{aligned} (\nabla_Z A)(x, Y, \eta) := & Z(A(x, Y, \eta)) - A(\nabla_Z X, Y, \eta) \\ & - A(x, \nabla_Z Y, \eta) - A(x, Y, \nabla_Z^\perp \eta) \end{aligned}$$

Thm: ("Constraint Equations" for isometric immersions)

The following equations hold for $M \subseteq (\bar{M}, \bar{g})$:

$\forall X, Y, Z, W \in T(TM), \forall \eta, \zeta \in T(NM)$. we have

Gauss: $\bar{R}(\underset{T}{\underset{T}{X}}, \underset{T}{\underset{T}{Y}}, \underset{T}{\underset{T}{Z}}, \underset{T}{\underset{T}{W}}) = R(x, Y, Z, W) - \langle A(Y, W), A(X, Z) \rangle + \langle A(X, W), A(Y, Z) \rangle$

Ricci: $\bar{R}(\underset{N}{\underset{N}{X}}, \underset{N}{\underset{N}{Y}}, \underset{N}{\underset{N}{\eta}}, \underset{N}{\underset{N}{\zeta}}) = \langle R^\perp(x, Y)\eta, \zeta \rangle + \langle \underbrace{[S_\eta, S_\zeta]}_{\text{Commutator } S_\eta \circ S_\zeta - S_\zeta \circ S_\eta}(x), Y \rangle$

Codazzi: $\bar{R}(\underset{T}{\underset{T}{X}}, \underset{T}{\underset{T}{Y}}, \underset{T}{\underset{T}{Z}}, \underset{T}{\underset{T}{\eta}}) = (\nabla_Y A)(x, Z, \eta) - (\nabla_X A)(Y, Z, \eta)$

Idea of Proof: \exists orthogonal splitting $T\bar{M} = TM \oplus NM$

Proof: Recall: $\bar{\nabla}_x Y = \nabla_x Y + A(x, Y)$

∇ : tangent to M

A : normal to M

$$\begin{aligned}
 \bar{R}(x, Y) Z &:= \bar{\nabla}_Y \bar{\nabla}_x Z - \bar{\nabla}_x \bar{\nabla}_Y Z + \bar{\nabla}_{[x, Y]} Z \\
 &= \bar{\nabla}_Y (\nabla_x Z + A(x, Z)) - \bar{\nabla}_x (\nabla_Y Z + A(Y, Z)) \\
 &\quad + \nabla_{[x, Y]} Z + A([x, Y], Z) \\
 &= \nabla_Y \nabla_x Z + A(Y, \nabla_x Z) + \bar{\nabla}_Y (A(x, Z)) \\
 &\quad - \nabla_x \nabla_Y Z - A(x, \nabla_Y Z) - \bar{\nabla}_x (A(Y, Z)) \\
 &\quad + \nabla_{[x, Y]} Z + A([x, Y], Z) \\
 &= R(x, Y) Z + A(Y, \nabla_x Z) - A(x, \nabla_Y Z) + A([x, Y], Z) \\
 &\quad + \bar{\nabla}_Y (A(x, Z)) - \bar{\nabla}_x (A(Y, Z)).
 \end{aligned}$$

Taking inner product with a tangential $W \in T(TM)$,

$$\begin{aligned}
 \bar{R}(x, Y, Z, W) &= R(x, Y, Z, W) + \langle \bar{\nabla}_Y (A(x, Z)), W \rangle - \langle \bar{\nabla}_x (A(Y, Z)), W \rangle \\
 &= R(x, Y, Z, W) - \langle A(x, Z), \bar{\nabla}_Y W \rangle + \langle A(Y, Z), \bar{\nabla}_x W \rangle \\
 &= R(x, Y, Z, W) - \langle A(x, Z), (\bar{\nabla}_Y W)^{\perp} \rangle + \langle A(Y, Z), (\bar{\nabla}_x W)^{\perp} \rangle \\
 &= R(x, Y, Z, W) - \langle A(x, Z), A(Y, W) \rangle + \langle A(Y, Z), A(x, W) \rangle
 \end{aligned}$$

Gauss!

Taking inner product with a normal $\eta \in T(NM)$,

$$\begin{aligned}
 \bar{R}(x, Y, Z, \eta) &= A(Y, \nabla_x Z, \eta) - A(x, \nabla_Y Z, \eta) + A([x, Y], Z, \eta) \\
 &\quad + \langle \bar{\nabla}_Y (A(x, Z)), \eta \rangle - \langle \bar{\nabla}_x (A(Y, Z)), \eta \rangle
 \end{aligned}$$

$$\left(\begin{aligned} \text{Note: } & \langle \bar{\nabla}_Y(A(x, z)), \eta \rangle \\ & = Y(A(x, z, \eta)) - \langle A(x, z), (\bar{\nabla}_Y \eta)^{\mathbb{N}} \rangle \\ & = Y(A(x, z, \eta)) - \langle A(x, z), \nabla_Y^\perp \eta \rangle \end{aligned} \right)$$

$$\begin{aligned} \bar{R}(x, Y, z, \eta) &= \underline{A(Y, \nabla_x z, \eta)} - \underline{A(x, \nabla_Y z, \eta)} + \underline{A(\nabla_x Y - \nabla_Y x, z, \eta)} \\ &\quad + \underline{Y(A(x, z, \eta))} - \underline{A(x, z, \nabla_Y^\perp \eta)} \\ &\quad - \underline{X(A(Y, z, \eta))} + \underline{A(Y, z, \nabla_x^\perp \eta)} \\ &= \underline{(\nabla_Y A)(x, z, \eta)} - \underline{(\nabla_x A)(Y, z, \eta)} \end{aligned}$$

Codazzi!

$$\text{Recall: } \bar{\nabla}_x \eta = -S_\eta(x) + \nabla_x^\perp \eta$$

$$\begin{aligned} \bar{R}(x, Y) \eta &= \bar{\nabla}_Y \bar{\nabla}_x \eta - \bar{\nabla}_x \bar{\nabla}_Y \eta + \bar{\nabla}_{[x, Y]} \eta \\ &= \bar{\nabla}_Y (-S_\eta(x) + \nabla_x^\perp \eta) - \bar{\nabla}_x (-S_\eta(Y) + \nabla_Y^\perp \eta) \\ &\quad - S_\eta([x, Y]) + \nabla_{[x, Y]}^\perp \eta \\ &= \text{"Tangential terms"} \\ &\quad - [\bar{\nabla}_Y(S_\eta(x))]^{\mathbb{N}} + [\bar{\nabla}_x(S_\eta(Y))]^{\mathbb{N}} \\ &\quad + \nabla_Y^\perp \nabla_x^\perp \eta - \nabla_x^\perp \nabla_Y^\perp \eta + \nabla_{[x, Y]}^\perp \eta \end{aligned}$$

Taking inner product with a normal $\zeta \in \mathcal{T}(\text{NM})$.

$$\begin{aligned} \bar{R}(x, Y, \eta, \zeta) &= \langle R^\perp(x, Y) \eta, \zeta \rangle \\ &= \langle [\bar{\nabla}_Y(S_\eta(x))]^{\mathbb{N}}, \zeta \rangle + \langle [\bar{\nabla}_x(S_\eta(Y))]^{\mathbb{N}}, \zeta \rangle \end{aligned}$$

Note: $\langle [\bar{\nabla}_Y(S_\eta(x))]^\vee, \zeta \rangle = \langle \bar{\nabla}_Y(S_\eta(x)), \zeta \rangle$

$$= Y(\langle S_\eta(x), \zeta \rangle) - \langle S_\eta(x), (\bar{\nabla}_Y \zeta)^T \rangle$$

$$= \langle S_\eta(x), S_\zeta(Y) \rangle$$

$$\bar{R}(x, Y, \eta, \zeta) = \langle R^\perp(x, Y)\eta, \zeta \rangle + \langle S_\eta(x), S_\zeta(Y) \rangle - \langle S_\eta(Y), S_\zeta(x) \rangle$$

Ricci!

$$= \langle R^\perp(x, Y)\eta, \zeta \rangle + \langle (S_\zeta \circ S_\eta - S_\zeta \circ S_\eta)(x), Y \rangle$$

Several Remarks

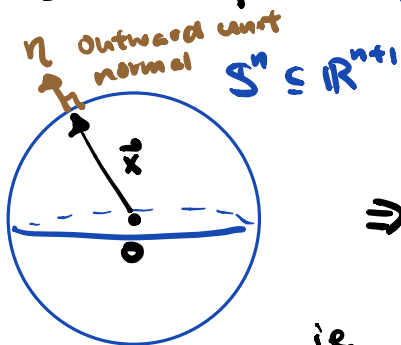
(1) In codimension 1 (ie. hypersurface), the Ricci equation is trivial.

Reason: $\text{codim } 1 \Leftrightarrow NM$ is 1-dim'l $\zeta = f\eta$

Ricci: $\bar{R}(x, Y, \eta, \eta) = \langle R^\perp(x, Y)\eta, \eta \rangle + \langle [S_\eta \circ S_\eta](x), Y \rangle$

(2) The unit sphere $S^n \subseteq (\mathbb{R}^{n+1}, \bar{g}_{\text{Eucl}})$ has $K \equiv 1$.

Pf:



$\eta = \bar{x}$ position vector

$$\Rightarrow \bar{\nabla}_x \eta = \bar{\nabla}_x \bar{x} = X \Rightarrow S_\eta(x) = -X$$

ie. $A_\eta(x, Y) = \langle S_\eta(x), Y \rangle = -\langle X, Y \rangle$

Gauss: $\bar{R}(x, y, z, w) = R(x, y, z, w) - \langle A(y, w), A(x, z) \rangle + \langle A(x, w), A(y, z) \rangle$

$\equiv 0$
 $\because \mathbb{R}^{n+1}$ is flat

$$\Rightarrow R^{S^n}(x, y, z, w) = \langle y, w \rangle \langle x, z \rangle - \langle x, w \rangle \langle y, z \rangle$$

ie $K^{S^n} \equiv 1$.

(iii) If $M^{n-1} \subset (\bar{M}^n, \bar{g})$ codim 1 and (\bar{M}, \bar{g}) has constant sectional curvature, then for $\eta =$ unit normal (locally)

$$\text{Codazzi eq}^1 \Leftrightarrow \nabla_x (S_\eta(y)) - \nabla_y (S_\eta(x)) = S_\eta([x, y])$$

(Ex: Prove this.)